Standing capillary-gravity waves of finite amplitude: Corrigendum

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The uniqueness condition that was utilized by the author (Concus 1962) is considered. The condition, which excludes certain fluid depths, is shown to be physically unacceptable because it is essentially impossible to satisfy in practice. The resulting invalidation of the perturbation method is discussed, and a revision is presented, which invokes the presence of viscosity and allows retention of the previously obtained solutions. The revision may also be applied to the work of other authors who utilized the same method to solve other standing-wave problems.

Consider the form that the uniqueness condition (12) takes for $\delta = 0$. It requires that the mean fluid depth h be such that, for every integer n and j

$$\frac{n \tanh nh}{\tanh h} \neq j^2 \quad (n = 2, 3, \ldots; j = 1, 2, \ldots). \tag{12a}$$

The manner in which these excluded depths are distributed over the positive real line $(0, \infty)$ is of importance and may be determined as follows. For any $n > 1$, the ratio $\tanh nh/\tanh h$ is continuous and decreases monotonically from n to 1 as h increases from 0 to ∞. Thus, for each n there are approximately $n - \sqrt{n}$ distinct values of the fluid depth excluded by (12a). The totality of all such excluded depths obtained by letting n run through its range of positive integers will form a denumerably infinite set of points ranging over the positive real line $(0, \infty)$.

It is next shown that this set is densely distributed over $(0, \infty)$. Let h_1 and h_2 be any two values of h such that $0 < h_1 < h_2$, and consider the interval (h_1, h_2). As h increases through all the values in (h_1, h_2), the ratio $n \tanh nh/\tanh h$ decreases through all the values in the interval

$$(n \tanh nh_2/\tanh h_2, \quad n \tanh nh_1/\tanh h_1). \tag{40}$$

Consider the interval

$$(\tanh nh_2/\tanh h_2, \quad \tanh nh_1/\tanh h_1). \tag{41}$$

Since $\tanh h_1 < \tanh h_2$ and $\tanh nh_2 < 1$ and since an N can be found such that

$$\tanh nh_1 > \tfrac{1}{2} + \tfrac{1}{2}(\tanh h_1/\tanh h_2) \quad \text{for all} \quad n \geqq N,$$

the interval (41) contains the smaller interval

$$\left(\frac{1}{\tanh h_2}, \frac{1}{2}\left[\frac{1}{\tanh h_1} + \frac{1}{\tanh h_2}\right]\right)$$

for all $n \geqq N$.† Choose a rational number p/q in this interval. Then (40) contains the rational number np/q. Choose $n = N^2pq$. Then $n \tanh nh/\tanh h$ takes on the value of the perfect square N^2p^2 for some h in the interval (h_1, h_2). Hence, any such interval (h_1, h_2) contains at least one value of h excluded by (12a). Therefore the values of h excluded by (12a) form a denumerably infinite set which is densely distributed over the entire positive real line $(0, \infty)$. The proof carries through in the same manner for the depths excluded by (12) with $\delta \neq 0$ in the same manner, and the result is identical.

Because the set of excluded depths is everywhere dense in the interval $(0, \infty)$, a serious difficulty arises in applying the solution to a physical situation. When one measures a fluid depth in practice, one does not do so exactly, but specifies some small interval of non-zero length in which the depth is known to lie. However because the set of excluded depths is everywhere dense, one must always have an excluded value of h (in fact a countable infinity of them) in this interval. It is therefore impossible to satisfy the uniqueness condition in a given physical situation.

More serious than the inability to satisfy the uniqueness condition, which the denseness of the excluded depths implies, is the inability to deal with the resonances that also were thought to be eliminated by the uniqueness condition. For example, in solving (22), the uniqueness condition was utilized to prevent the appearance of a resonant, or secular, term, which would be unbounded in time and violate the periodicity condition (7). For the higher-order equations, other such secular terms can appear, each of them corresponding to a fluid depth excluded by (12). In such conditions of resonance and for nearby depths, which yield conditions of near-resonance, a higher-order term can become so large as to be of comparable magnitude with a lower-order one. In such a case, the derivation of the perturbation equations must be modified to take this fact into account. (For an example of the modifications required for one such resonant term, see Mack 1962.) The previous analysis has shown, moreover, that for any fluid depth there are infinitely many such resonant or near-resonant terms with which one must contend. Thus, there are always too many high-order terms which cannot be neglected in the development, because they are of comparable magnitude with the low-order terms. Such considerations make it appear that the expansion procedure is not a suitable one for finding the desired periodic motion.

Attempts of the author to find another method of developing the perturbation solution for inviscid periodic motion that will not fall into the above difficulty have not yet been successful, nor has a proof for the existence of the desired non-linear periodic motion been obtained. Experimental evidence of Fultz (1962), however, has verified that the use of the expansion procedure under question gives results that are physically observable.

Some justification of the procedure on theoretical grounds can be made by invoking the presence of viscosity. It is only for the lower frequencies that one is justified in using an inviscid model for predicting the approximate behaviour of fluids of low viscosity. For higher frequencies (larger n) the model becomes

† The author is indebted to Dr G. Nooney for suggesting this and other features of the proof.

inaccurate, and the appearance of the high-frequency waves predicted by the model will be inhibited or at least their amplitudes diminished in magnitude by the viscous forces that are present. One, therefore, must consider the possibility of a higher-order term in the perturbation expansion being of comparable magnitude with the fundamental term only for the smaller values of n. For larger n, the physical situation is no longer described by the mathematical formulation, and the high-frequency waves will not be present in the amounts predicted by the theory.

Equation (12) may be utilized, therefore, only for the smaller values of n to ensure uniqueness and to predict the depths for which there may exist a resonant coupling with the fundamental linear mode. For larger values of n, however, both the expansion and (12) lose meaning. A cut-off value for n can be determined by taking the smallest n for which the viscous decay time of the nth linear mode, a quantity proportional to $1/n^2$, is less than the period of the fundamental mode. For the case considered on p. 576 with $h = 0{\cdot}25$, $\delta = 0{\cdot}02$, and $\lambda = 10$ cm in water, the corresponding cut-off value for n is 4.

The above remarks can also be applied to the work of Tadjbakhsh & Keller (1960), where this method of developing the perturbation solution was first introduced, to the work of Verma & Keller (1962) and Moiseyev (1958), where similar uniqueness conditions have also appeared, and, as well, to the work of Penny & Price (1952) and of Mack (1962).

Two other errors should be corrected:

Page 569. At the beginning of the next-to-last line of the first paragraph before

$$`\epsilon k^{-1}\eta(x,t)\ldots,\text{' insert '}\epsilon = ka \text{ the expansion parameter'.}$$

Page 571. In the line following (23), change '(21)' to '(19)'.

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REFERENCES

Concus, Paul 1962 Standing capillary-gravity waves of finite amplitude. *J. Fluid Mech.* **14**, 568–76.

Fultz, Dave 1962 An experimental note on finite-amplitude standing gravity waves. *J. Fluid Mech.* **13**, 193–212.

Mack, Lawrence R. 1962 Periodic finite-amplitude, axisymmetric gravity waves. *J. Geophys. Res.* **67**, 829–43.

Moiseyev, N. N. 1958 On the theory of non-linear vibrations of a liquid of finite volume. *J. Appl. Math. Mech.* **22**, 860–72 (translated from *Prikl. Mat. Mec.* **22**, 612–21).

Penney, W. G. & Price, A. I. 1952 Finite periodic stationary gravity waves in a perfect liquid. *Phil. Trans.* A, **244**, 254–84.

Tadjbakhsh, I. & Keller, J. B. 1960 Standing surface waves of finite amplitude. *J. Fluid Mech.* **8**, 442–51.

Verma, G. R. & Keller, J. B. 1962 Three-dimensional standing surface waves of finite amplitude, *Phys. Fluids*, **5**, 52–6.